INF280: Competitive programming

Basic graph traversals and shortest paths

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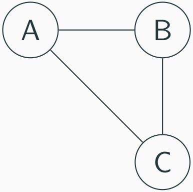
You all know graphs:

- Set of nodes N
- Set of edges $E \subseteq N \times N$
- Edges can be undirected or directed, i.e., $(a, b) \neq (b, a)$



$N \quad \{A, B, C\}$
$E \quad \{(A, B), (A, C), (B, C)\}$

Data Structures

Several options to represent graphs:

- Adjacency matrix:
  - bool G[MAXN][MAXN];
  - G[x][y] is true if an edge between node x and y exists
  - Replace bool by int to represent weighted edges
- Adjacency list:
  - vector<int> Adj[MAXN];
  - y is in Adj[x] if an edge between node x and y exists
  - Pairs to represent weights
- Edge list:
  - vector<pair<int, int> > Edges;
  - Edges contains a pair of nodes if an edge exists between them
- Nodes and edges may also be custom structs or classes

Simple Traversals

Simple Traversals

Depth-First Search

Depth-First Search

Visit each node in the graph once:

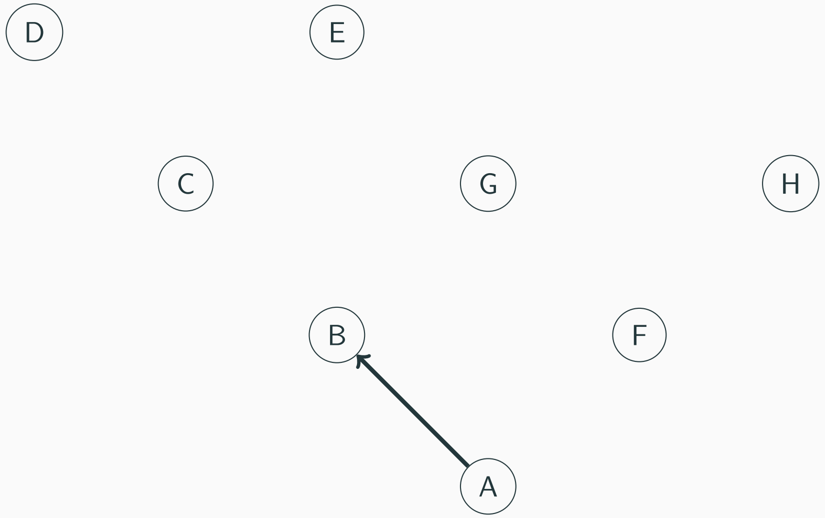
- Recursive implementation below
- Manage stack yourself for iterative version
- First visit child nodes then siblings

```
int state[ID_NODE_MAX] ;
const int NOT_VISITED = 0, IN_VISIT = 1 , VISITED = 2 ;
void dfs(int node) {
  if(state[node] == NOT_VISITED) {
    state[node] = IN_VISIT ;
    for(auto v : nxt[node])
      dfs(v);
    state[node] = VISITED ;
  }
}
```

Applications of DFS

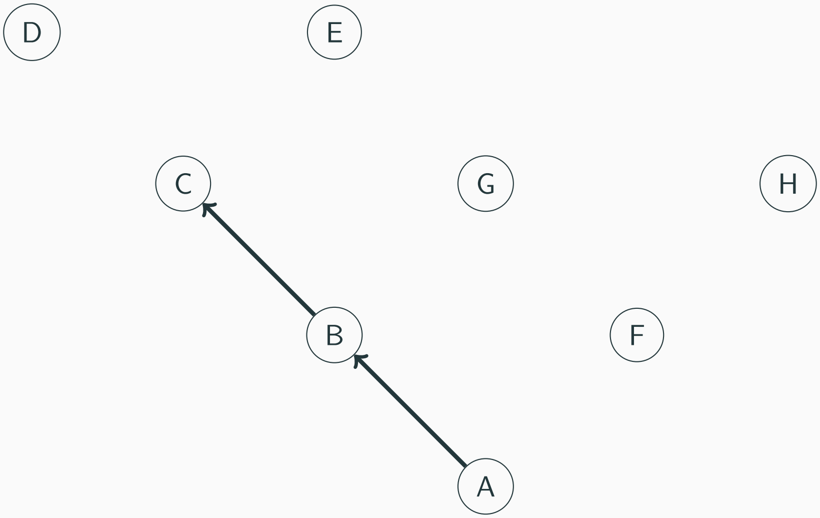
- Determine a topological order of nodes
- Detect if a cycle exists
- Check reachability between nodes
- Decompose graph into connected components
- Decompose graph in strongly connected components
- Examples: `https://visualgo.net/dfsbfs`

Tarjan representation of DFS



Useful to understand what happens...

Tarjan representation of DFS



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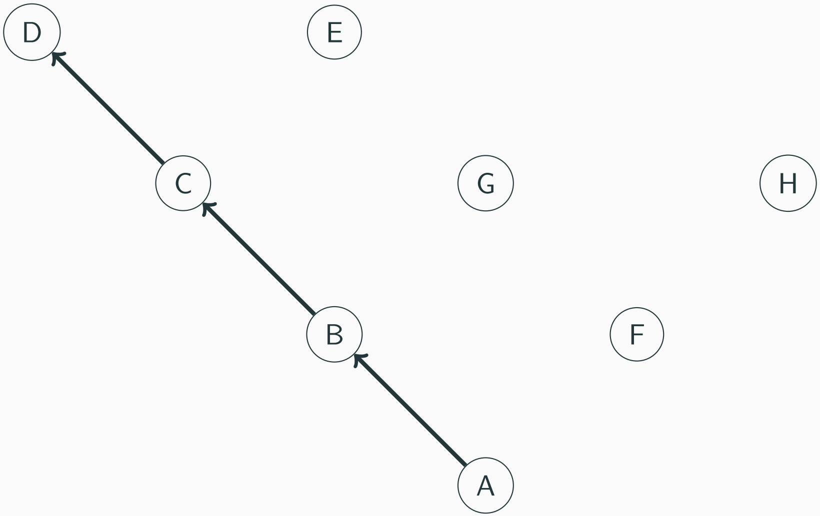

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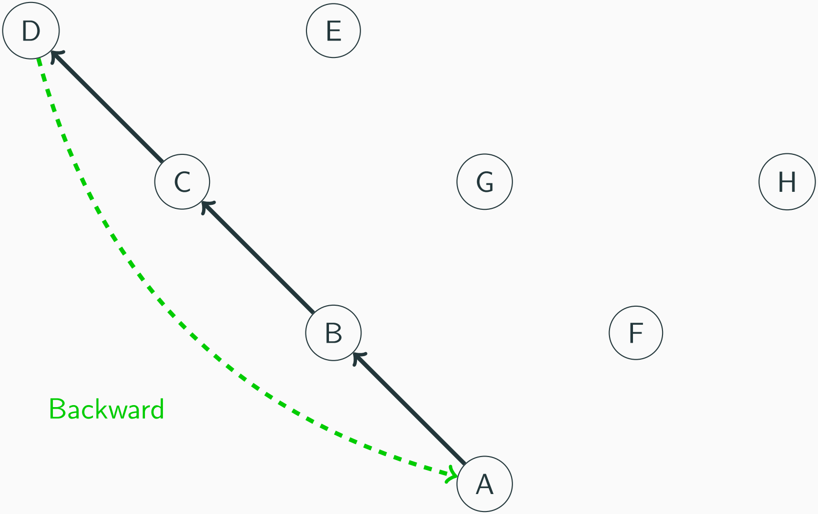

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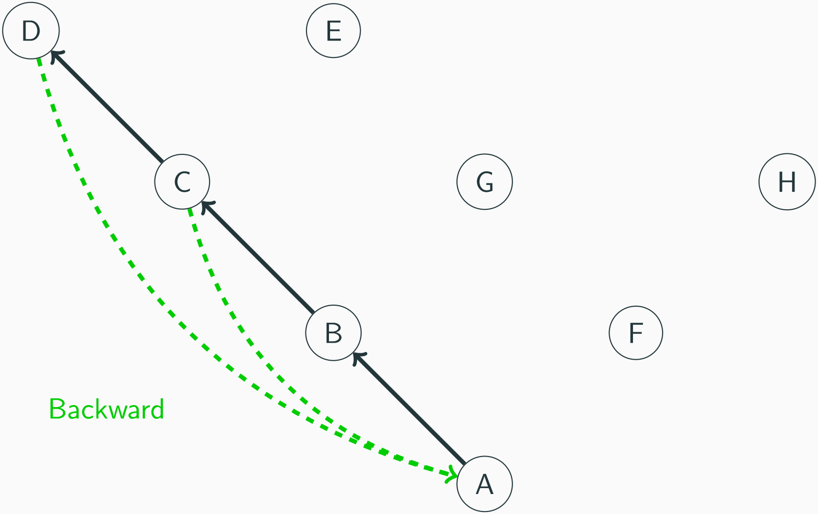

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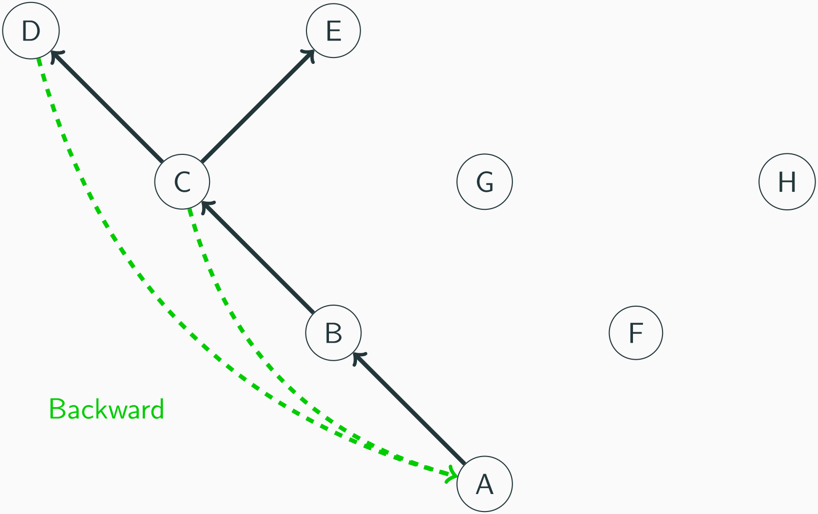

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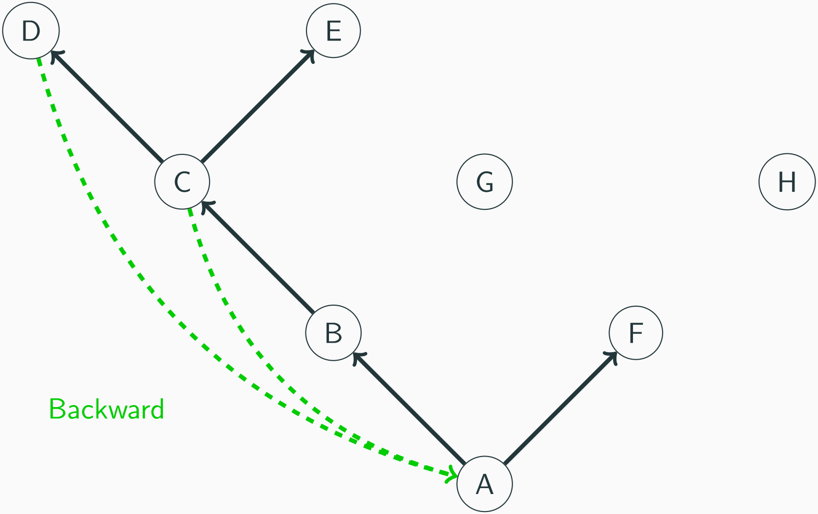

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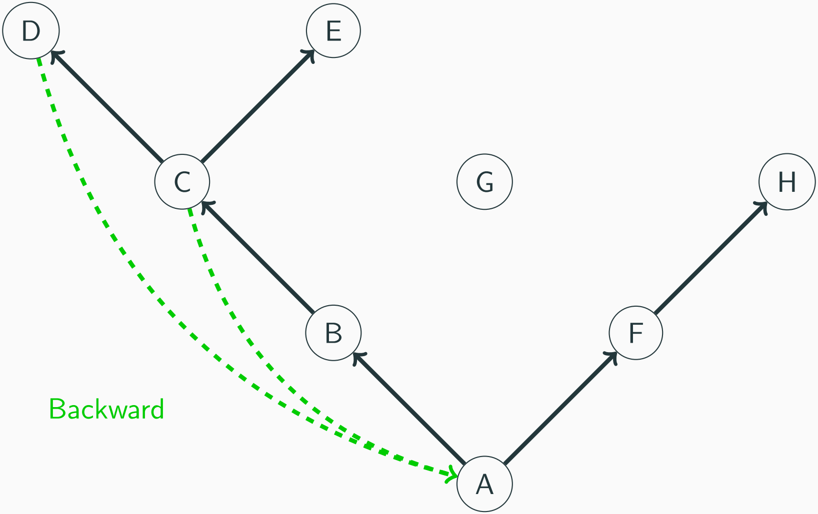

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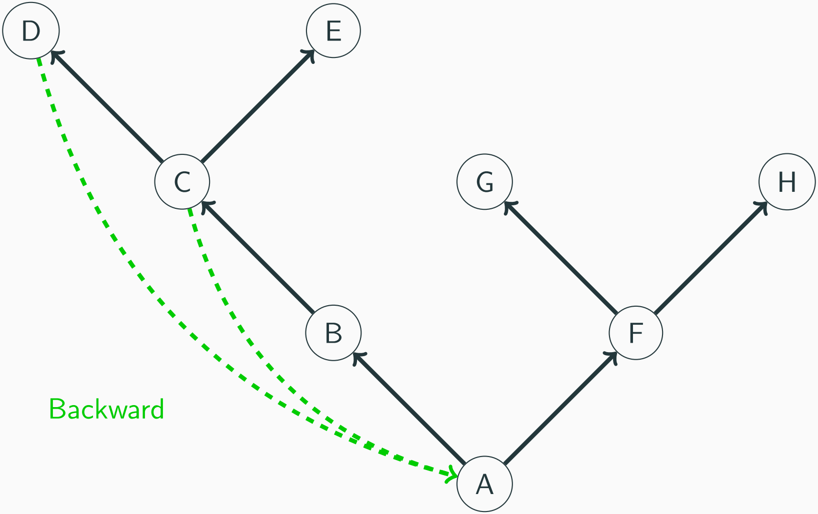

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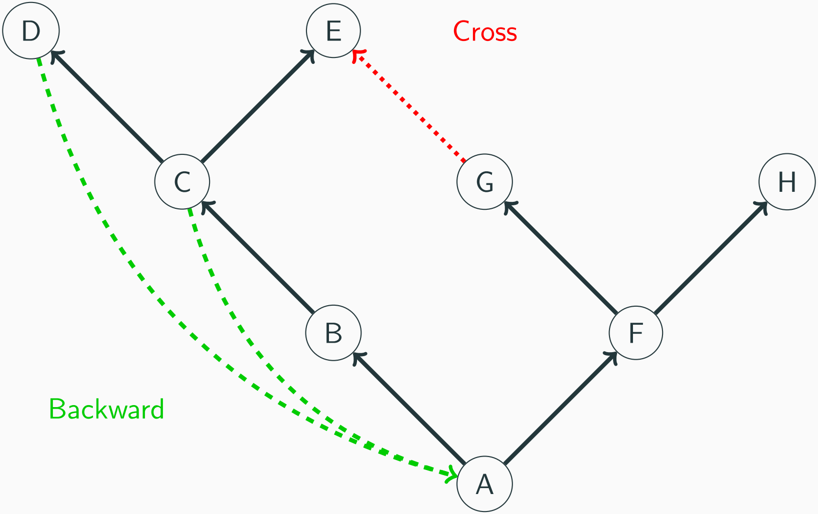

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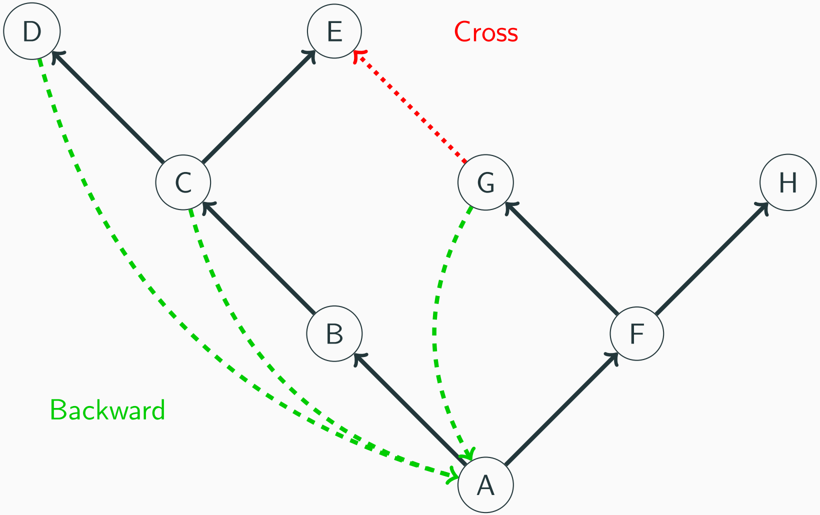

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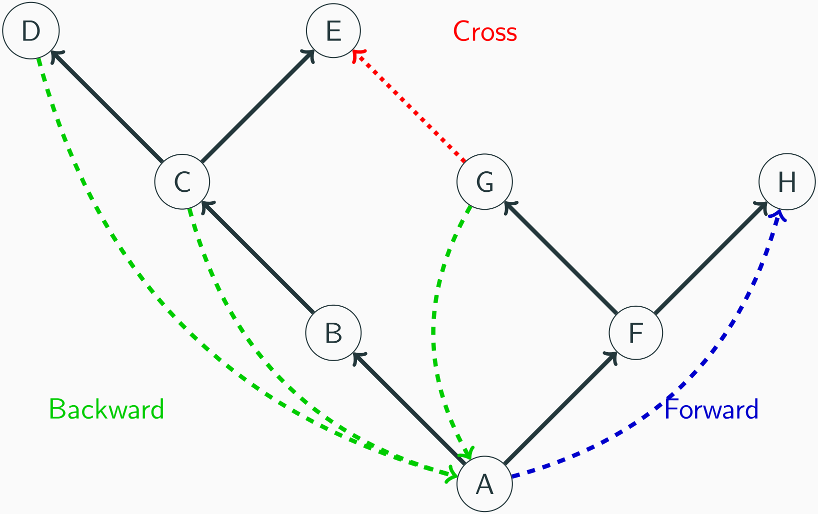

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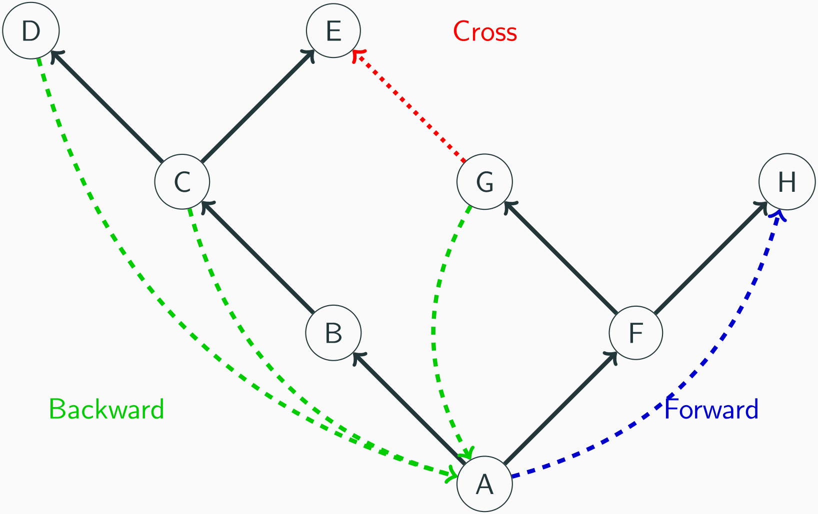

Exercise: compute Strongly Connected Component

Simple Traversals

Breadth-First Search

Breadth-First Search

Visit each node in the graph once:

- Similar to DFS, but replaces stack by queue

```
int seen[NB_NODE_MAX] ;
void bfs(int start) {
  vector<int> todo = {start} ;
  seen[start] = true ;
  for(size_t id = 0 ; id < todo.size() ; id++)
    for(auto v : nxt[todo[id]])
      if(!seen[v]) {
        seen[v] = true;
        todo.push_back(v);
      }
}
```

Applications of BFS

- Shortest path search
  - Stop processing when a given node d was found
  - Minimizes number of hops, i.e., all edges have same weight or 0-1 Weights
  - Generalization follows next
- Examples: `https://visualgo.net/dfsbfs`

Simple Traversals

0-1 Breadth-First Search

Breadth-First Search with edges of bounded distance

```
vector<int> nodes_at[MAX_DISTANCE];
void bfs(int start) {
  fill(dist,dist+NB_NODES_MAX,INF);
  nodes_at[0] = {start} ;
  dist[start] = 0 ;
  for(int cur_dist = 0 ; cur_dist < MAX_DISTANCE ; cur_dist++ )
    for(size_t id = 0 ; id < nodes_at[cur_dist].size() ; id++) {
      const int node = nodes_at[cur_dist][id] ;
      if(dist[node] == cur_dist)
        for(auto [neigh,len] : nxt[node])
          if(dist[neigh] > cur_dist+len) {
            dist[neigh] = cur_dist+len ;
            nodes_at[dist[neigh]].push_back(neigh);
          }
    }
}
```

Finding Paths

Finding Paths

Dijkstra

Dijkstra

- Dijkstra's algorithm generalizes BFS
- Constraint: all edges need to have non-negative weights
- Use a priority queue to choose which node to examine next

Finding Paths

Bellman-Ford

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Bellman-Ford DP problem: "$q(n, k)$ is the minimal distance of n from the source node using k intermediate edges"

Bellman-Ford can also be seen as a way to solve a linear system with inequalities of the form: $x_i + c_i \leq y_i$

Bellman-Ford Algorithm

```
int from[MAX_NB_EDGES], to[MAX_NB_EDGES],weight[MAX_NB_EDGES];
int dist[MAX_PATH_LENGTH+1][MAX_NB_NODES];
bool BellmanFord(int root) {
  fill(dist[0],dist[MAX_PATH_LENGTH],INF);
  dist[0][root] = 0;
  for(int len = 0 ; len < MAX_PATH_LENGTH ; len++)
    for (int e = 0 ; e < nb_edges ; e++)
      dist[len+1][to[e]] = min(dist[len+1][to[e]],
                               dist[len][from[e]]+weight[e]);
  // to be explained later; check for negative cycles
  return dist[MAX_PATH_LENGTH][target];
}
```

- replace `dist[l][n]` with `dist[n]` $= min_l($`dist[l][n]`$)$
- `MAX_PATH_LENGTH` is at most `nb_nodes` long

Bellman-Ford Algorithm

```
int dist[MAX_NB_NODES];
void BellmanFord(int root, int target) {
  fill(dist, dist+MAX_NB_NODES, INF);
  dist[root] = 0;
  for(int k = 0 ; k < nb_nodes - 1 ; k++)  // N - 1 times
    for (int i = 0 ; i < nb_edges ; i++)
      dist[to[i]] = min(dist[to[i]], dist[from[i]]+weight[i]);
}
```

Bellman-Ford Algorithm

```
bool detect_negative_cycle_BellmanFord(int root, int target) {
  fill(dist, dist+MAX_NB_NODES, INF);
  dist[root] = 0;
  for(int k = 0 ; k < nb_nodes - 1 ; k++)  // N - 1 times
    for (int i = 0 ; i < nb_edges ; i++)
      dist[to[i]] = min(dist[to[i]], dist[from[i]]+weight[i]);
 // now time to check for negative cycles:
  int dist_target = dist[target]; // copy distance
  for(int k = 0 ; k < nb_nodes - 1 ; k++)  // N - 1 times
    for (int i = 0 ; i < nb_edges ; i++)
      dist[to[i]] = min(dist[to[i]], dist[from[i]]+weight[i]);
  return dist[target] < dist_target ; // negative cycle?
}
```

Finding Paths

Floyd-Warshall

Floyd-Warshall

- Dijkstra and Bellman-Ford compute shortest paths
  - From a single source (root)
  - To all other (reachable) nodes
  - This is known as: single-source shortest path problem
- Floyd-Warshall extends this to compute the shortest paths between all pairs of nodes
- This is known as: all-pairs shortest path problem

Floyd-Warshall

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Floyd-Warshall answers the DP problem: "q(start,end,pivot): what is the shortest path between `start` and `end` going through intermediate nodes 1..`pivot`?"

Floyd-Warshall Algorithm

```
int dist[MAX_NB_NODES][MAX_NB_NODES];
// We store q(start,end,pivot) in dist[start][end]
void FloydWarshall() {
  // initialize distance
  fill(dist[0],dist[MAX_NB_NODES],INF);
  for (int e = 0 ; e < nb_edges ; e++)
    dist[fr[e]][to[e]] = min(dist[fr[e]][to[e]], weight[e]);
  // now compute
  for(int pivot = 0 ; pivot < nb_nodes ; pivot++)
    for(int start = 0 ; start < nb_nodes ; start++)
      for(int end = 0 ; end < nb_nodes ; end++)
        dist[start][end] = min(dist[start][end],
                     dist[start][pivot]+dist[pivot][end]);
}
// WARNING, the order of the loops is important!!!
// for french speakers Pivot Début Fin => PDF algorithm
```

Finding Paths

Improvements

Keeping track of the path

We only considered the length of the path so far:

- All of the above algorithms can track the actual shortest path
- This allows to *print* each edge/node along the path
- Basic idea:
  - Introduce an array `int Predecessor[MAXN]`
    (Use two-dimensional array for Floyd-Warshall)
  - Updated whenever `Dist[v]` changes
  - Simply set to the new predecessor `u`

Heuristics – A* Search

Heuristics may speed-up the path search

- Bellman-Ford and Floyd-Warshall equally explore all possibilities
- Dijkstra *prefers* nodes with shorter distance
- Basic idea behind A* Search:
  - Extension to Dijkstra's algorithm
  - Introduce an estimation of the remaining distance
  - Prefer nodes with minimal estimated *remaining* distance
- Advantages
  - May converge faster than Dijkstra
  - Can be used to compute approximate solutions (trading speed for precision)

Applications of DFS

Applications of DFS

Let us Recall:

- Determine a topological order of nodes
- Detect if a cycle exists
- Check reachability between nodes
- Decompose graph into connected components
- Decompose graph in strongly connected components
- Examples: `https://visualgo.net/dfsbfs`

Reachability

```
int state[NB_NODES_MAX] ;
const int NOT_VISITED = 0 , VISITED = 2 ;
bool rec_reachable(int node, int target) {
  if(state[node] == NOT_VISITED) {
    state[node] = VISITED ;
    for(auto v : nxt[node])
      if(rec_reachable(v,target))
        return true;
  }
  return false;
}

bool reachable(int source, int target) {
  fill(state,state+NB_NODES_MAX, NOT_VISITED);
  return rec_reachable(source,target);
}
```

Reachability alternative

```
int state[NB_NODES_MAX] ;
const int NOT_VISITED = 0 , VISITED = 2 ;
int rec_reachable(int node) {
  if(state[node] == VISITED)
    return 0;
  state[node] = VISITED ;
  int newly_discovered = 1 ;
  for(auto v : nxt[node])
    newly_discovered += rec_reachable(v) ;
  return newly_discovered;
}

int nb_reachable(int source) {
  fill(state,state+NB_NODES_MAX, NOT_VISITED);
  return rec_reachable(source);
}
```

Topological order of nodes / detecting of cycles

```
int state[NB_NODES_MAX] ;
const int NOT_VISITED = 0, IN_VISIT = 1 , VISITED = 2 ;

int order[NB_NODES_MAX] ; // edges (a,b) are such that order[a]>order[b
int cur_order = 0 ;

void topological_order(int node) {
  if(state[node] == IN_VISIT) {
    // DO SOMETHING, A CYCLE EXISTS !!!
  }
  if(state[node] == NOT_VISITED) {
    state[node] = IN_VISIT ;
    for(auto v : nxt[node])
      topological_order(v);
    order[node] = cur_order++; // mark the order
    state[node] = VISITED ;
  }
}
```

Decompose into connected component (undirected)

```
void dfs_decompose(int node, int comp_id) {
  if(component[node] < 0) {
    component[node]=comp_id;
    for(auto v : nxt[node])
      dfs_decompose(v,comp_id);
  }
}

void decompose( ) {  // total O(V+E)
  fill(component,component+NB_NODES_MAX,-1);
  for(int n = 0 ; n < nb_nodes ; n++)
    dfs_decompose(n,n);
}
```

Decompose into connected component (undirected)

```
void dfs_decompose(int node, int comp_id) {
  if(component[node] < 0) {
    component[node]=comp_id;
    for(auto v : nxt[node])
      dfs_decompose(v,comp_id);
  }
}
void decompose2( ) {
  fill(component,component+NB_NODES_MAX,-1);
  int nb_comp = 0;
  for(int n = 0 ; n < nb_nodes ; n++) //
    if(component[n] < 0 )
      dfs_decompose(n,nb_comp++);
}
```

We will see more graph algorithms next week...